

RELEVANT SAMPLING IN FINITELY GENERATED SHIFT-INVARIANT SPACES

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ABSTRACT. We consider random sampling in finitely generated shift-invariant spaces $V(\Phi) \subset L^2(\mathbb{R}^n)$ generated by a vector $\Phi = (\varphi_1, \dots, \varphi_r) \in L^2(\mathbb{R}^n)^r$. Following the approach introduced by Bass and Gröchenig, we consider certain relatively compact subsets $V_{R,\delta}(\Phi)$ of such a space, defined in terms of a concentration inequality with respect to a cube with side lengths R . Under very mild assumptions on the generators, we show that for R sufficiently large, taking $O(R^n \log(R^{n^2/\alpha'}))$ many random samples (taken independently uniformly distributed within C_R) yields a sampling set for $V_{R,\delta}(\Phi)$ with high probability. Here $\alpha' \leq n$ is a suitable constant. We give explicit estimates of all involved constants in terms of the generators $\varphi_1, \dots, \varphi_r$.

1. INTRODUCTION

Digital signal processing rests on two basic operations: sampling and reconstruction. Sampling is the task of transforming the analog into a digital signal. The converse process is the reconstruction of the analog signal from the digital signal. However, these problem cannot be solved without extra information or assumptions on the analog signal under consideration. There exist several well-understood ways of formulating these restrictions, eg. by assuming the analog signal f to be bandlimited, or more generally, that it belongs to a shift-invariant space [1, 2, 3, 5, 6, 10, 11, 15, 16, 18, 19, 20]. Bandlimited signals of finite energy are completely characterized by their regular samples if they are taken at a sufficiently high rate (Nyquist criterion), as described by the famous classical Shannon sampling theorem. A more general class of such spaces are finitely generated shift-invariant spaces of the following type

$$V(\Phi) := \left\{ \sum_{k \in \mathbb{Z}^n} C^T(k) \Phi(\cdot - k) : C \in (\ell^2)^r \right\}$$

where $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_r)^T$ for $\varphi_i \in L^2(\mathbb{R}^n)$ ($i = 1, 2, \dots, r$ and $1 \leq p \leq \infty$) is the so-called generator of $V(\Phi)$ and $C = \{c^1, c^2, \dots, c^r\}$ with $\|C\|_{(\ell^2)^r}^2 = \sum_{i=1}^r \|c^i\|_{\ell^2}^2$.

In the past years, the random sampling method has been commonly used in the field of compressed sensing [8, 10] and image processing[7]. The general context of learning from random sampling has been studied by Cucker, Smale, Zhou, et al. (see [9, 17]). Recently,

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the random sampling problems were studied by Bass and Gröchenig in the multivariate trigonometric polynomials spaces [4] and bandlimited functions spaces [5, 6]. Yang and Wei discussed the problem when some randomly chosen samples $X = \{x_j : j \in J\}$ forms a set of sampling in the shift-invariant space [20].

Random sampling has become a rather active area of research. However, so far, most results deal with functions defined on compact domains. For functions defined on \mathbb{R}^n , however, one is faced with the dilemma of choosing a proper probability distribution for the sampling set, as there is no uniform distribution on all of \mathbb{R}^n . As a remedy to this problem, Bass and Gröchenig introduced the notion of *relevant sampling* in [5, 6]. Here, the random sampling sets are confined to a fixed compact subset K . As a tradeoff, the sampling results are not intended to hold for all elements of the space under consideration, but only for those functions who are concentrated (in a suitable sense) within K .

Following the approach taken by Bass and Gröchenig [4, 5, 6], we restrict attention to the subset

$$V_{R,\delta}(\Phi) := \left\{ f \in V(\Phi) : \int_{C_R} |f(x)|^2 dx \geq (1 - \delta) \int_{\mathbb{R}^n} |f(x)|^2 dx \right\},$$

where $C_R = [-R/2, R/2]^d$ and $0 < \delta < 1$. Thus $V_{R,\delta}(\Phi)$ is the subset of $V(\Phi)$ consisting of those functions whose energy is largely concentrated on C_R .

We are looking for conditions on random sets X satisfying following inequalities:

$$(1) \quad c\|f\|_{L^2} \leq \left(\sum_{x_j \in X} |f(x_j)|^2 \right)^{\frac{1}{2}} \leq C\|f\|_{L^2}.$$

In this paper, we pursue and extend the approach of [6] to a rather general setting of finitely generated shift-invariant spaces, with very mild conditions on the generators. While the overall proof strategy could be largely preserved, the details of the arguments in [6] often relied on the specific, well-understood setting of prolate spheroidal wave functions, and the adaptation to the general case was not straightforward. In any case, we believe that the subsequent results and arguments provide an interesting contrast and supplement to [6].

The paper is organized as follows. In Section 2, we introduce the main result and conditions on the generators. In Section 3, the localization operator associated to $V(\Phi)$ and C_R is described and proved. In Section 4, we discuss the random sampling in finite sums of eigenspaces. At the end, the proof of main result is presented in Section 5.

2. STATEMENT OF THE MAIN RESULT

Throughout the paper, we consider a finitely generated shift-invariant subspace $V(\Phi) \subset L^2(\mathbb{R}^n)$, defined as the closed linear span of a tuple of generators $(\varphi_1, \dots, \varphi_r) \in L^2(\mathbb{R}^n)^r$, shifted by the integers. We assume that the associated system $(T_m \varphi_i)_{i=1, \dots, r, m \in \mathbb{Z}^d}$ is a frame for $V(\Phi)$. Furthermore, we fix a dual frame obtained as integer shifts of the vectors $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_r \in L^2(\mathbb{R}^n)$.

Thus we obtain for all $f \in V(\Phi)$, that

$$f = \sum_{m,i} \langle f, T_m \tilde{\varphi}_i \rangle T_m \varphi_i$$

In the following, we will mostly work with the notation

$$(2) \quad P_\Phi = \sum_{m,i} (T_m \varphi_i) \otimes (T_m \tilde{\varphi}_i) ,$$

where unconditional convergence in the strong operator topology is guaranteed by the frame properties, for all $f \in L^2(\mathbb{R}^n)$. The tensor product notation $v \otimes w$ refers to the rank-one operator $(v \otimes w)f \mapsto \langle f, w \rangle v$. Our assumptions show that P_Φ is the identity on $V(\Phi)$, and since $(T_k \tilde{\varphi}_i)_{k,i}$ also span $V(\Phi)$, the kernel of P_Φ is $V(\Phi)^\perp$; thus P_Φ is the orthogonal projection onto $V(\Phi)$.

We let $\|x\|_\infty = \max(|x_1|, \dots, |x_n|)$. Given $R > 0$, we write $Q_R : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ for the orthogonal projection operator $f \mapsto f \cdot \chi_{C_R}$.

2.1. Assumptions and chief result. We collect our assumptions on the generators, with the associated constants, in the following list:

(A.0) [**Bessel Constants**] The upper frame constants for the systems $(T_m \varphi_i)_{m,i}$ and $(T_m \tilde{\varphi}_i)_{m,i}$ are denoted by C_0 and \tilde{C}_0 , respectively. Hence, for all $f \in L^2(\mathbb{R}^n)$:

$$\sum_{m,i} |\langle f, T_m \varphi_i \rangle|^2 \leq C_0 \|f\|^2$$

and

$$\sum_{m,i} |\langle f, T_m \tilde{\varphi}_i \rangle|^2 \leq \tilde{C}_0 \|f\|^2 .$$

(A.1) [**Reproducing Kernel**] The point evaluations are bounded linear functionals on $V(\Phi)$. Hence, using the Fischer-Riesz theorem, there exists a family $(v_x)_{x \in \mathbb{R}^d} \subset V(\Phi)$ satisfying $f(x) = \langle f, v_x \rangle$, for all $f \in V(\Phi)$. We assume that

$$C_1 = C_1(\Phi) = \sup_x \|v_x\|_2 < \infty .$$

This implies in particular $\|f\|_\infty \leq C_1 \|f\|_2$, for all $f \in V(\Phi)$.

(A.2) [**Plancherel-Polya-type inequality**] There exists a constant $C_2 = C_2(\Phi)$ such that for every subset $\Gamma \subset \mathbb{R}^n$ with covering index

$$N_0(\Gamma) = \max_{k \in \mathbb{Z}^n} \text{card}(\Gamma \cap (k + [-1/2, 1/2]^n))$$

and every $f \in V(\Phi)$, we have

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 \leq C_2 N_0(\Gamma) \|f\|_2^2 .$$

(A.3) [**Decay property**] There exists $\alpha > 0$ and $C_3 = C_3(\Phi)$, such that for all $i = 1, \dots, k$: $\|\varphi_i \cdot (1 - \chi_{C_R})\|_2^2 \leq C_3 R^{-\alpha}$.

Clearly, the decay property is fulfilled by any vector of compactly supported functions.

Remark 1. Observe that the requirements are rather mild. They are in fact fulfilled by the n -dimensional sinc function φ and the space $V(\varphi)$ of bandlimited functions: The shifts of the sinc function provide an ONB, in particular we may take $\tilde{\varphi} = \varphi$. We therefore get $C_0 = \tilde{C}_0 = 1$. In addition, the sinc function acts as a reproducing convolution kernel for $V(\varphi)$, thus (A.1) holds with $C_1 = 1$. The Plancherel-Polya constant for $V(\varphi)$ was explicitly computed as $C_2 = e^{n\pi}$ in the appendix of [6]. The localization property holds with $C_3 = n$ and $\alpha = 1$. Hence the following result indeed provides a generalization of the main theorem [6], although with less sharp constants, and a slightly worse sampling rate: Instead of $O(R^n \log(R^n))$, we obtain $O(R^n \log(R^{n^2/\alpha'}))$. \square

Throughout this paper, we will repeatedly refer to the constants $\alpha' = \min(n, \alpha)$, for α from assumption (A.3), and

$$(3) \quad \beta = 3 + 2 \sqrt[n]{2^{n+2} r C_0 \tilde{C}_0^2 C_3}.$$

The main result of this paper is the following generalization of [6, Theorem 1]:

Theorem 2. *Assume that the frame generators fulfill assumptions (A.0)-(A.3). Let $(x_j)_{j \in \mathbb{N}}$ denote a sequence of independent random variables, each uniformly distributed in C_R . Let*

$$R_0 = \max \left(1, \sqrt[n]{2 C_3}, \sqrt[n]{C_1^2} \right).$$

Let $R \geq R_0$, and assume that $\delta, \nu \in (0, 1)$ are sufficiently small to guarantee that

$$\frac{\nu^2}{C_1^2(1 + \nu/3)} \leq 3 \log 3 - 2 \text{ and } A = \frac{1}{2} - \delta - \nu - 12\delta C_2 > 0.$$

Let $0 < \epsilon < 1$. If the number s of samples satisfies

$$(4) \quad s \geq R^n \frac{1 + \nu/3}{\nu^2} \log \frac{2\beta^n R^{n^2/\alpha'}}{\epsilon},$$

then the sampling inequality

$$(5) \quad \forall f \in V_{R,\delta}(\Phi) : A s R^{-n} \|f\|_2^2 \leq \sum_{j=1}^s |f(x_j)|^2 \leq s \|f\|_2^2$$

holds with probability at least $1 - \epsilon$.

2.2. Generators fulfilling the assumptions of Theorem 2. As will be seen shortly, our main result applies to large classes of generators. For the formulation of the following result, recall the definition of the Wiener amalgam spaces: Given a function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, we define its Wiener amalgam norm, for $1 \leq p < \infty$, via

$$\|f\|_{W(L^p)}^p = \sum_{k \in \mathbb{Z}^n} \text{ess sup}_{x \in [0,1]^n} |f(x + k)|^p,$$

and denote the space of all continuous f for which this norm is finite by $W_0(L^p)$. It has been noted in [2] that Wiener amalgam norms are useful tools for the study of sampling problems, and the following results provide further evidence for this principle.

Proposition 3. *Assume that $\Phi \in (W_0(L^1))^r$. Then assumptions (A.1) and (A.2) are fulfilled.*

Proof. At least formally, the reproducing kernel can be obtained in a straightforward way from (2): For all $f \in V(\Phi)$, we have

$$\begin{aligned} f(x) &= \sum_{k,i} \langle f, T_m \tilde{\varphi}_i \rangle T_m \varphi_i(x) \\ &= \sum_{k,i} \int_{\mathbb{R}^n} f(y) \overline{\tilde{\varphi}_i(y-k)} dy \varphi_i(x-k) \\ &= \int_{\mathbb{R}^n} f(y) \overline{v_x(y)} dy, \end{aligned}$$

where

$$v_x(y) = \sum_{k,i} \tilde{\varphi}_i(y-k) \overline{\varphi_i(x-k)}.$$

Now [3] establishes that the sum actually converges and yields a square-integrable v_x . Furthermore, the fact that the translates of Φ are Bessel sequence allow to conclude that

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \|v_x\|_2^2 &\leq C_0 \sup_{x \in \mathbb{R}^n} \|(\varphi_i(x-k))_{i,k}\|_{\ell^2(\mathbb{Z}^n)^r}^2 \\ &= C_0 \sup_{x \in [0,1]^n} \sum_{i,k} |\varphi_i(x-k)|^2 \\ &\leq C_0 \sum_{i,k} \sup_{x \in [0,1]^n} |\varphi_i(x-k)|^2 \\ &= C_0 \sum_i \|\varphi_i\|_{W(L^2)}^2 \\ &\leq C_0 \sum_i \|\varphi_i\|_{W(L^1)}^2 < \infty, \end{aligned}$$

using the norm-decreasing inclusion $W_0(L^1) \subset W_0(L^2)$. This yields assumption (A.1) with constant

$$C_1 = \left(C_0 \sum_{i=1}^r \|\varphi_i\|_1^2 \right)^{1/2}.$$

For the proof of the Plancherel-Polya inequality, first note that it is sufficient to consider sets $\Gamma \subset \mathbb{N}$ with $N_0(\Gamma) = 1$; the more general statement then follows from writing a set Γ with $N_0(\Gamma) = K$ as the union of K sets with density one.

Given $f \in V(\Phi)$, define $\text{osc}(f) : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ as

$$\text{osc}(f)(x) = \sup_{\|y\|_\infty \leq 1/2} |f(x) - f(x+y)|.$$

By [3] there exists a constant $M > 0$ such that

$$\forall f \in V(\Phi) : \|\text{osc} f\|_2 \leq M \|f\|_2.$$

Now let $\Lambda = \{k \in \mathbb{Z}^n : \Gamma \cap k + [-1/2, 1/2]^n \neq \emptyset\}$. By essential disjointness of the shifted cubes, we have that

$$\sum_{k \in \Lambda} \int_{k + [-1/2, 1/2]^n} |f(x)|^2 dx \leq \|f\|_2^2.$$

We can relate this sum to $\sum_{\gamma \in \Gamma} |f(\gamma)|^2$ as follows: For each $\gamma \in \Gamma$ pick a $k_\gamma \in \Lambda$ such that $\gamma \in k_\gamma + [-1/2, 1/2]^n$. k_γ is uniquely determined, and by assumption on Γ , we get that $\Gamma \ni \gamma \mapsto k_\gamma$ is one-to-one. We then get the following series of estimates:

$$\begin{aligned} \sum_{\gamma \in \Gamma} \left| |f(\gamma)|^2 - \int_{k_\gamma + [-1/2, 1/2]^n} |f(x)|^2 dx \right| &\leq \sum_{\gamma \in \Gamma} \left| \int_{k_\gamma + [-1/2, 1/2]^n} |f(\gamma)|^2 - |f(x)|^2 dx \right| \\ &\leq \sum_{\gamma \in \Gamma} \int_{k_\gamma + [-1/2, 1/2]^n} ||f(\gamma)|^2 - |f(x)|^2| dx \\ &\leq \sum_{\gamma \in \Gamma} \int_{k_\gamma + [-1/2, 1/2]^n} |f(\gamma) - f(x)| (|f(\gamma)| + |f(x)|) dx \\ &\leq \int_{\mathbb{R}^n} \text{osc} f(x) (2|f(x)| + \text{osc} f(x)) dx \\ &\leq \|\text{osc} f\|_2 (2\|f\|_2 + \|\text{osc} f\|_2) \\ &\leq M(M+2)\|f\|_2^2 \end{aligned}$$

But this implies

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 \leq M(M+2)\|f\|_2^2 + \sum_{k \in \Lambda} \int_{k + [-1/2, 1/2]^n} |f(x)|^2 dx \leq (M+1)^2 \|f\|_2^2,$$

and the Plancherel-Polya inequality is established. \square

We thus obtain easily checked criteria in terms of continuity and moderate decay:

Corollary 4. *Assume that Φ is vector of functions generating a frame under shifts, and consisting of continuous functions satisfying the decay estimate*

$$\forall i = 1, \dots, r : |\varphi_i(x)| \leq C(1 + \|x\|_\infty)^{-n-\epsilon},$$

for some $\epsilon > 0$. Then conditions (A.1) through (A.3) are fulfilled, with $\alpha = n$.

Proof. The decay estimate implies $\Phi \in (W_0(L^1))^r$, and thus (A.1) and (A.2) follow from the previous proposition. (A.3), with $\alpha = n$, is easily verified. \square

3. THE LOCALIZATION OPERATOR ASSOCIATED TO $V(\Phi)$ AND C_R

Throughout the remainder of the paper, we will always assume that Φ is a set of frame generators fulfilling assumptions (A.0) through (A.3). The proof strategy for our main result is an adaptation of the method devised by Bass and Gröchenig [6] for the special case of bandlimited functions. We introduce the localization operator

$$A_R = P_\Phi \circ Q_R \circ P_\Phi.$$

We will show that A_R is a selfadjoint Hilbert-Schmidt operator, and therefore has a basis of eigenvectors with associated square-summable spectrum. Denoting by \mathcal{P}_N the projection onto the span of the eigenvectors associated to the largest N eigenvalues, we then establish a random sampling theorem for this space. We then show how sampling of the elements in $V_{R,\delta}(\Phi)$ can be related to sampling in \mathcal{P}_N . The proper choice of N , which will allow to transfer the random sampling result to $V_{R,\delta}(\Phi)$, depends on certain estimates concerning the decay of the spectrum of A_R .

Note that the following result is valid for all shift-generated frames of closed subspaces of $L^2(\mathbb{R}^n)$, without any further assumptions on the generators.

Lemma 5. *A_R is a positive-semidefinite Hilbert-Schmidt operator.*

Proof. Positive semidefiniteness of A_R follows from

$$\langle A_R f, f \rangle = \langle Q_R P_\Phi f, P_\Phi f \rangle = \|Q_R P_\Phi f\|_2^2,$$

using that Q_R is a selfadjoint projection.

We next show that $Q_R P_\Phi$ is Hilbert-Schmidt, which will imply that A_R is Hilbert-Schmidt, as a composition of a bounded and a Hilbert-Schmidt operator.

For $(j, m) \in \{1, \dots, k\} \times \mathbb{Z}^n$, we have

$$Q_R \circ (T_m \varphi_j) \otimes (T_m \tilde{\varphi}_j) = (Q_R T_m \varphi_j) \otimes (T_m \tilde{\varphi}_j).$$

Using (2) and boundedness of Q_R , we can write

$$Q_R P_\Phi = \sum_{j,m} (Q_R T_m \varphi_j) \otimes (T_m \tilde{\varphi}_j)$$

with unconditional convergence in the strong operator topology.

We next prove that

$$(6) \quad \sum_{j,m} \|Q_R T_m \varphi_j\|_2^2 < \infty.$$

For fixed $k \in \mathbb{Z}$ with $k > R$ and $m_1, m_2 \in k\mathbb{Z}^n$ with $m_1 \neq m_2$, the sets $C_R + m_1$ and $C_R + m_2$ are disjoint. It follows for arbitrary $x \in \mathbb{R}^d$

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} \|Q_R T_{km+x} \varphi_j\|_2^2 &= \sum_{m \in \mathbb{Z}^n} \int_{C_R} |\varphi_j(y - x - km)|^2 dy \\ &= \sum_{m \in \mathbb{Z}^n} \int_{C_R + km} |\varphi_j(y - x)|^2 dy \leq \|\varphi_j\|_2^2, \end{aligned}$$

by the above observed disjointness. Since \mathbb{Z}^n can be covered by k^n cosets of $k\mathbb{Z}^n$, we obtain

$$\sum_{m,j} \|Q_R T_m \varphi_j\|_2^2 < \infty.$$

Thus, with $\vartheta_{m,j} = Q_R T_m \varphi_j$, we have

$$Q_R P_\Phi = \sum_{m,j} \vartheta_{m,j} \otimes (T_m \tilde{\varphi}_j) , \quad \sum_{m,j} \|\vartheta_{m,j}\|_2^2 < \infty ,$$

and hence

$$(7) \quad (Q_R P_\Phi)(Q_R P_\Phi)^* = \sum_{m,j} (Q_R P_\Phi T_m \tilde{\varphi}_j) \otimes \vartheta_{m,j} = \sum_{m,j} (Q_R T_m \tilde{\varphi}_j) \otimes \vartheta_{m,j}$$

with convergence in the strong operator topology. By the same argument as for equation (6), we see that

$$\sum_{m,j} \|Q_R T_m \tilde{\varphi}_j\|_2^2 < \infty .$$

This observation, in combination with (6) and the Cauchy-Schwarz inequality, yields that

$$\sum_{m,j} \text{trace}(|(Q_R T_m \tilde{\varphi}_j) \otimes \vartheta_{m,j}|) = \sum_{m,j} \|Q_R T_m \tilde{\varphi}_j\| \|\vartheta_{m,j}\| < \infty .$$

Hence we see that the expansion (7) in fact converges in the trace class norm as well, finally implying that $Q_R P_\Phi$ is Hilbert-Schmidt. \square

It follows that A_R has an ONB of eigenvectors. Denoting the nonzero eigenvalues by $(\lambda_n)_{n \in I}$ and the associated eigenfunctions by $(\psi_n)_{n \in I}$, with I being either \mathbb{N} or $\{1, \dots, M\}$ for some integer M , A_R is given as the sum

$$\sum_{n \in I} \lambda_n \psi_n \otimes \psi_n .$$

As the equation $\psi_n = \lambda_n^{-1} P_\Phi^* Q_R P_\Phi \psi_n$ shows, we have $\psi_n \in V(\Phi)$, and thus $P_\Phi(\psi_n) = \psi_n$. Since A_R is Hilbert-Schmidt, we have $\sum_n |\lambda_n|^2 < \infty$, and since it is positive-semidefinite, we may assume $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$. Furthermore, since A_R is a composition of projections, we have $\lambda_1 \leq \|A_R\|_{op} \leq 1$. We let

$$\mathcal{P}_N = \text{span}\{\psi_n : n = 1, \dots, N\} .$$

For the space of bandlimited functions, the eigenfunctions are the well-known prolate spheroidal wave functions introduced by Slepian and Pollak [14]. For the sampling results derived below, some information on the spectrum of A_R is necessary. We let

$$N(R) = \max\{n \in \mathbb{N} : \lambda_n \geq 1/2\} ,$$

and $N(R) = 0$ whenever $\lambda_1 < 1/2$. Thus, whenever $N(R) > 0$, then $\lambda_{N(R)} \geq 1/2 > \lambda_{N(R)+1}$.

The following lemma provides an estimate for $N(R)$, derived from the decay assumptions on the generators.

Lemma 6. *Let β be defined by (3). Then for all $R > \max(1, \sqrt[3]{2C_3})$, the inequalities $0 < N(R) \leq \beta^n R^{n^2/\alpha'}$ hold.*

Proof. We use the well-known minimax formula

$$\lambda_m = \inf \{ \sup \{ \langle A_R f, f \rangle : f \perp \mathcal{H}, \|f\|_2 = 1 \} : \mathcal{H} \subset L^2(\mathbb{R}^n), \dim(\mathcal{H}) \leq m \} .$$

Now fix $S > R$, and consider

$$\mathcal{H}_S = \text{span} \{ T_m \tilde{\varphi}_i : 1 \leq i \leq r, \|m\|_\infty \leq S/2 \} .$$

It follows that $\dim(\mathcal{H}_S) = (2\lfloor S/2 \rfloor + 1)^n \leq (\lfloor S \rfloor + 1)^n$. Next assume that $f \in \mathcal{H}_S^\perp$ is a unit vector. Then we obtain

$$\langle A_R f, f \rangle = \sum_{m,j,m',j'} c_{m,j} a_{m,j,m',j'} \overline{c_{m',j'}} ,$$

with $c_{m,i} = \langle f, T_m \tilde{\varphi}_i \rangle$, and an infinite, positive semidefinite matrix

$$\mathcal{A} = (a_{m,j,m',j'})_{((m,j),(m',j')) \in (\mathbb{Z}^n \times \{1, \dots, r\})^2}$$

defined by

$$a_{m,j,m',j'} = \begin{cases} \langle Q_R T_m \varphi_j, T_{m'} \varphi_{j'} \rangle & \min(\|m\|_\infty, \|m'\|_\infty) > S/2 \\ 0 & \text{otherwise} \end{cases} ,$$

recall the assumption $f \perp T_m \tilde{\varphi}_i$, for $\|m\|_\infty \leq S/2$. In particular, we get

$$\langle A_R f, f \rangle \leq \|c\|_2^2 \|\mathcal{A}\|_{\text{op}} \leq \tilde{C}_0 \|\mathcal{A}\|_{\text{op}}$$

We estimate the Hilbert-Schmidt norm of the matrix, as follows:

$$\begin{aligned} \|\mathcal{A}\|_{HS}^2 &= \sum_{m,j,m',j', \|m\|_\infty \geq S, \|m'\|_\infty > S/2} |a_{m,j,m',j'}|^2 \\ &\leq \sum_{m,j, \|m\|_\infty > S/2} \sum_{m',j'} |\langle Q_R T_m \varphi_j, T_{m'} \varphi_{j'} \rangle|^2 \\ &\leq \sum_{m,j, \|m\|_\infty > S/2S} C_0 \|Q_R T_m \varphi_j\|_2^2 . \end{aligned}$$

We can now employ a similar reasoning as in the proof of Lemma 5: Picking $k = \lfloor R \rfloor + 1$, we have for arbitrary distinct $\ell, \ell' \in \{0, \dots, k-1\}^n$ that $\ell + C_R \cap \ell' + C_R$ has measure zero. Furthermore, $m + C_R \cap C_{S/2-R}$ has measure zero, whenever $\|m\|_\infty > S/2$. This implies that

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n, \|m\|_\infty > S/2} \|Q_R T_m \varphi_j\|_2^2 &= \sum_{\ell \in \{0, \dots, k-1\}^n} \sum_{m \in \ell + k\mathbb{Z}^n, \|m\|_\infty > S/2} \|Q_R T_m \varphi_j\|_2^2 \\ &= \sum_{\ell \in \{0, \dots, k-1\}^n} \sum_{m \in \ell + k\mathbb{Z}^n, \|m\|_\infty > S/2} \int_{C_R + m} |\varphi_j(x)|^2 dx \\ &\leq \sum_{\ell \in \{0, \dots, k-1\}^n} \int_{\mathbb{R}^n \setminus C_{S/2-R}} |\varphi_j(x)|^2 dx \\ &\leq C_3 (\lfloor R \rfloor + 1)^n (S/2 - R)^{-\alpha} , \end{aligned}$$

using the decay assumption (A.3). Thus we arrive at

$$\langle A_R f, f \rangle^2 \leq \tilde{C}_0^2 \|\mathcal{A}\|_{\text{op}}^2 \leq \tilde{C}_0^2 \|\mathcal{A}\|_{HS}^2 \leq r \tilde{C}_0^2 C_0 C_3 (\lfloor R \rfloor + 1)^n (S/2 - R)^{-\alpha}$$

For $S = (\beta - 1)R^{n/\alpha'}$ with β according to (3), one finds that $R > 1$ and the definition of α' yield that $R \leq R^{n/\alpha'}$, and hence

$$\begin{aligned} r\tilde{C}_0^2 C_0 C_3 (S/2 - R)^{-\alpha} (\lfloor R \rfloor + 1)^n &\leq r\tilde{C}_0^2 C_0 C_3 \left(\frac{\beta - 3}{2} \right)^{-\alpha} R^{-n} (\lfloor R \rfloor + 1)^n \\ &\leq r\tilde{C}_0^2 C_0 C_3 2^n \left(\frac{\beta - 3}{2} \right)^{-\alpha} \\ &\leq \frac{1}{4}, \end{aligned}$$

by definition of β . Hence we have shown

$$\langle A_R f, f \rangle^2 < 1/4$$

for all unit vectors $f \in \mathcal{H}_S^\perp$.

If $R > 1$, then $N = (\lfloor (\beta - 1)R^{n/\alpha'} \rfloor + 1)^n \leq \beta^n R^{n^2/\alpha'}$, and the minimax estimate yields $\lambda_N < 1/2$. On the other hand, (A.3) implies

$$\lambda_1 \geq \langle A_R \varphi_i, \varphi_i \rangle \geq 1 - C_3 R^{-\alpha} > 1/2$$

as soon as $R > \sqrt[3]{2C_3}$. Hence we find for all $R > \max(1, \sqrt[3]{2C_3})$ that

$$0 < N(R) \leq \beta^n R^{n^2/\alpha'}.$$

□

Remark 7. The proof of Lemma 6 is the only place in the paper where we employ the decay assumption (A.3) on the generators. All subsequent estimates of $N(R)$ in the following depend on this result. The case of bandlimited functions provides one example where similar or sharper estimates may be available by alternative methods; here the estimate $\lfloor R \rfloor - 1 \leq N(R) \leq \lfloor R \rfloor + 1$ can be shown by Fourier-analytic arguments, see [12] for the one-dimensional case. This is the main reason for the suboptimal sampling rate $O(R^n \log R^{n^2/\alpha'})$ stated in Remark 1 for the bandlimited case. Using the estimate from [12] instead of Lemma 6, our subsequent arguments provide a sampling rate of $O(R^n \log R^n)$, just as in [6]. □

4. RANDOM SAMPLING IN FINITE SUMS OF EIGENSPACES

We continue our adaptation of [6]. Recall from the previous section that $\lambda_1 \geq \lambda_2 \geq \dots$ are the eigenvalues of A_R , with corresponding eigenfunctions ψ_1, ψ_2, \dots . The span of the first N eigenfunctions is denoted by \mathcal{P}_N . We let $\Delta_N = \text{diag}(\lambda_1, \dots, \lambda_N)$.

The aim of this section is to prove a random sampling statement for \mathcal{P}_N . It will follow by applying a matrix Bernstein inequality (stated in the following Theorem 8), which uses the following notation: For $A \in \mathbb{C}^{N \times N}$, we let $\|A\|$ denote the operator norm with respect to the euclidean norm. Further, the inequality $A \leq B$ for two matrices A, B of equal size means that $B - A$ is positive semidefinite.

Theorem 8. [13] *Let X_j be a sequence of independent, random self-adjoint $N \times N$ -matrices. Suppose that $\mathbb{E}X_j = 0$ and $\|X_j\| \leq B$ a.s. And let $\sigma^2 = \|\sum_{i=1}^s \mathbb{E}(X_j^2)\|$. Then for all $t > 0$,*

$$\mathbb{P}(\lambda_{\max}(\sum_{i=1}^s X_j) \geq t) \leq N \exp(-\frac{t^2/2}{\sigma^2 + Bt/3})$$

where $\lambda_{\max}(U)$ is the largest singular value of a matrix U so that $\|U\| = \lambda_{\max}(A^*A)^{1/2}$ is the operator norm.

The random matrices under consideration are constructed as follows: For each $j \in \mathbb{N}$ and $\ell \in \{1, \dots, k\}$, we introduce the $N \times N$ rank-one random matrix T_j^ℓ defined by

$$(8) \quad (T_j)_{k,l} = \psi_k(x_j) \overline{\psi_l(x_j)} .$$

Here the x_j denote i.i.d. random variables, uniformly distributed on C_R . We finally let

$$(9) \quad X_j = T_j - \mathbb{E}(T_j) .$$

The following provides useful estimates for the constants in the matrix Bernstein inequality. It is an analog of [6, Lemma 4].

Lemma 9. *Let X_j be defined via (8) and (9). Then the following hold:*

$$\begin{aligned} \mathbb{E}(X_j) &= 0 \quad , \quad \|X_j\| \leq \max(C_1^2, R^{-n}) \quad (a.s.), \\ \mathbb{E}(X_j^2) &\leq R^{-n} C_1^2 \Delta_N \quad , \quad \sigma^2 = \left\| \sum_{j=1}^s \mathbb{E}(X_j)^2 \right\| \leq s C_1^2 R^{-n} . \end{aligned}$$

Proof. It is obvious that $\mathbb{E}(X_j) = 0$. Since both T_j and $\mathbb{E}(T_j)$ are positive semidefinite, we have

$$\|X_j\| \leq \max(\|T_j\|, \|\mathbb{E}T_j\|) \leq \max(\|T_j\|, R^{-n}) .$$

Furthermore, recall from (10) that

$$\|T_j\| = \sup_{\|c\|_2=1} |\langle c, T_j c \rangle| = \sup_{f \in \mathcal{P}_N, \|f\|_2=1} |f(x_j)|^2 \leq C_1^2$$

using assumption (A.1) and $\mathcal{P}_N \subset V(\Phi)$.

We next compute

$$\mathbb{E}(X_j^2) = \mathbb{E}(T_j^2) - (\mathbb{E}(T_j))^2 = \mathbb{E}(T_j^2) - R^{-2n} \Delta_N^2 .$$

The square of the rank-one matrix T_j is computed as

$$T_j^2 = \left(\sum_{\ell=1}^N |\psi_\ell(x_j)|^2 \right) T_j .$$

Using the reproducing kernel $(v_x)_{x \in \mathbb{R}^n}$ for $V(\Phi)$, together with the fact that the eigenfunctions are an orthonormal system, we can estimate

$$\sum_{\ell=1}^N |\psi_\ell(x_j)|^2 = \sum_{\ell=1}^N |\langle \psi_\ell, v_{x_j} \rangle|^2 \leq \sum_{\ell=1}^{\infty} |\langle \psi_\ell, v_{x_j} \rangle|^2 \leq \|v_{x_j}\|_2^2 \leq C_1^2 .$$

Thus

$$T_j^2 \leq C_1^2 T_j ,$$

and since the expected value of a positive-semidefinite matrix valued random variable is positive semidefinite, we obtain

$$\mathbb{E}(T_j^2) \leq C_1^2 \mathbb{E}(T_j)$$

and thus

$$\mathbb{E}(X_j^2) \leq C_1^2 R^{-n} \Delta_N - R^{-2n} \Delta_N^2 .$$

Since the X_j are selfadjoint, X_j^2 is positive semidefinite, thus we have in fact proved

$$0 \leq \mathbb{E}(X_j^2) \leq C_1^2 R^{-n} \Delta_N$$

For positive semidefinite matrices, $A \leq B$ implies $\|A\| \leq \|B\|$, hence

$$\sigma^2 = \left\| \sum_{j=1}^s \mathbb{E}(X_j^2) \right\| \leq \left\| \sum_{j=1}^s C_1^2 R^{-n} \Delta_N \right\| \leq s C_1^2 R^{-n}$$

□

We can now formulate and prove a random sampling statement for \mathcal{P}_N .

Theorem 10. *Let $(x_j)_{j \in \mathbb{N}}$ denote a sequence of independent and identically distributed random variables, uniformly distributed in C_R . Assume that $R \geq \sqrt[n]{C_1^2}$. Then, for all $\nu \geq 0$ and $s \in \mathbb{N}$:*

$$\mathbb{P} \left(\inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{s} \sum_{j=1}^s (|f(x_j)|^2 - R^{-n} \|Q_R f\|_2^2) \leq R^{-n} \nu \right) \leq N \exp \left(- \frac{\nu^2 s}{C_1^2 R^n (1 + \nu/3)} \right) .$$

Proof. Let T_j be defined by (8). Using the fact that computing expectations amounts to integration over C_R (with respect to Lebesgue measure, normalized to one), in conjunction with $P_\Phi \psi_n = \psi_n$, one readily sees that

$$\begin{aligned} (\mathbb{E}(T_j))_{k,\ell} &= R^{-n} \int_{C_R} \psi_\ell(x) \overline{\psi_k(x)} dx \\ &= R^{-n} \langle Q_R \psi_\ell, \psi_k \rangle \\ &= R^{-n} \langle Q_R P_\Phi \psi_\ell, P_\Phi \psi_k \rangle \\ &= R^{-n} \langle A_R \psi_\ell, \psi_k \rangle \\ &= R^{-n} \lambda_\ell \delta_{\ell,k} \end{aligned}$$

and therefore

$$\mathbb{E}(T_j) = R^{-n} \Delta_N .$$

Furthermore, any unit vector $f \in \mathcal{P}_N$ is of the form $f = \sum_{n=1}^N c_n \psi_n$ with a unit vector $(c_n)_n$ of coefficients, and one obtains

$$(10) \quad |f(x_j)|^2 = \langle c, T_j c \rangle .$$

We thus have

$$\begin{aligned} & \inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{s} \sum_{j=1}^s (|f(x_j)|^2 - R^{-n} \|Q_R f\|_2^2) = \\ &= \inf_{c \in \mathbb{C}^N, \|c\|_2=1} \frac{1}{s} \sum_{j=1}^s \underbrace{(\langle c, T_j c \rangle - \langle c, \mathbb{E}(T_j) c \rangle)}_{=\langle c, X_j c \rangle} \\ &= \min \left\{ \rho : \rho \text{ eigenvalue of } \frac{1}{s} \sum_{j=1}^s X_j \right\} . \end{aligned}$$

Now the statement of the theorem follows from the Bernstein inequality for matrices formulated in Theorem 8, with proper constants provided by Lemma 9. \square

5. PROOF OF THE MAIN RESULT

It remains to transfer the random sampling statements from the spaces \mathcal{P}_N to the set $V_{R,\delta}(\Phi)$. The following lemma is a first step in this direction, by providing a norm estimate for the projection onto \mathcal{P}_N , for elements of $V_{R,\delta}(\Phi)$. It is an analog of [6, Lemma 5].

Lemma 11. *Let $N \in \mathbb{N}$, and let $\gamma \in \mathbb{R}$ with $\lambda_N \geq \gamma \geq \lambda_{N+1}$. Let E_N denote the orthogonal projection onto \mathcal{P}_N , and F_N denote the projection onto the orthogonal complement. Then for all $f \in V_{R,\delta}(\Phi)$, we have*

$$\begin{aligned} \|E_N f\|_2^2 &\geq \left(1 - \frac{\delta}{1-\gamma}\right) \|f\|_2^2 , \\ \|Q_R E_N f\|_2^2 &\geq \gamma \left(1 - \frac{\delta}{1-\gamma}\right) \|f\|_2^2 , \\ \|F_N f\|_2^2 &\leq \frac{\delta}{1-\gamma} \|f\|_2^2 . \end{aligned}$$

If $N = N(R) \neq 0$, these estimates simplify to

$$\begin{aligned} \|E_N f\|_2^2 &\geq (1 - 2\delta) \|f\|_2^2 , \\ \|Q_R E_N f\|_2^2 &\geq \left(\frac{1}{2} - \delta\right) \|f\|_2^2 , \\ \|F_N f\|_2^2 &\leq 2\delta \|f\|_2^2 . \end{aligned}$$

Proof. Let $f \in V_{R,\delta}(\Phi)$, w.l.o.g. $\|f\|_2 = 1$. Since $f = P_\Phi f$, we obtain

$$1 - \delta \leq \|Q_R f\|_2^2 = \|Q_R P_\Phi f\|_2^2 = \langle Q_R P_\Phi f, Q_R P_\Phi f \rangle = \langle A_R f, f \rangle = \sum_j |\langle f, \psi_j \rangle|^2 \lambda_j .$$

Let $c_j = \langle f, \psi_j \rangle$, and define

$$A = \|E_N f\|_2^2 = \sum_{j=1}^N |c_j|^2,$$

and $B = 1 - A = \|F_N f\|_2^2$. Then $\sum_{j=N+1}^{\infty} |c_j|^2 \leq \|F_N f\|_2^2 = 1 - A$. Using $\gamma \geq \lambda_{N+1} \geq \lambda_{N+2} > \dots$ and $\lambda_j \leq 1$, we find

$$\begin{aligned} A &= \sum_{j=1}^N |\langle f, \psi_j \rangle|^2 \geq \sum_{j=1}^N |\langle f, \psi_j \rangle|^2 \lambda_j \\ &= \sum_{j=1}^{\infty} |c_j|^2 \lambda_j - \sum_{j=N+1}^{\infty} |c_j|^2 \lambda_j \\ &\geq 1 - \delta - \gamma \left(\sum_{j=N+1}^{\infty} |c_j|^2 \right) \\ &\geq 1 - \delta - \gamma(1 - A). \end{aligned}$$

Solving this inequality for A yields $A \geq 1 - \frac{\delta}{1-\gamma}$, which implies $B \leq \frac{\delta}{1-\gamma}$. Finally, $\gamma \leq \lambda_N$ yields $\|Q_R E_N f\|_2^2 = \sum_{j=1}^N \lambda_j |c_j|^2 \geq \gamma A \geq \gamma(1 - \frac{\delta}{1-\gamma})$.

For $N = N(R) \neq 0$, we may pick $\gamma = 1/2$, which results in the estimates given for this case. \square

With the estimates from Lemma 11, the proof of the next lemma is a verbatim adaptation of the argument showing [6, Lemma 7], and therefore omitted.

Lemma 12. *Let $N \in \mathbb{N}$ and $\lambda_N \geq \gamma \geq \lambda_{N+1}$. Let $\{x_j : j = 1, \dots, s\} \subset C_R$ have covering index N_0 , and assume that the inequality*

$$\frac{1}{s} \sum_{j=1}^s (|p(x_j)|^2 - R^{-n} \|Q_R p\|_2^2) \geq -\nu R^{-n} \|f\|_2^2$$

holds for all $p \in \mathcal{P}_N$. Then the inequality

$$(11) \quad \sum_{j=1}^s |f(x_j)|^2 \geq A \|f\|_2^2$$

holds for all $f \in V_{R,\delta}(\Phi)$, with the constant

$$A = \frac{s}{R^n} \left(\gamma - \frac{\gamma\delta}{1-\gamma} - \nu \right) - 2C_2 N_0 \frac{\delta}{1-\gamma}.$$

A further ingredient is the following tail estimate for the covering number of a random set.

Lemma 13. [6, Lemma 8] *Suppose $R \geq 2$ and $\{x_j : j = 1, \dots, s\}$ are independent and identically distributed random variables that are uniformly distributed over C_R . Let $a > R^{-n}$. Then*

$$\mathbb{P}(N_0 > as) \leq (R+2)^n \exp(-s(a \log(aR^n) - (a - R^{-n}))),$$

where $N_0 = \max_{k \in \mathbb{Z}^n} \text{card}(\{x_j\} \cap (k + [-1/2, 1/2]^n))$.

Theorem 14. *Let $(x_j)_{j \in \mathbb{N}}$ be a sequence of independent random variables, each uniformly distributed in C_R . Assume that $R \geq \max(1, \sqrt[3]{2C_3}, \sqrt[3]{C_1^2})$, and furthermore*

$$\delta < \frac{1}{2(1+12C_2)}, \nu < \frac{1}{2} - \delta(1+12C_2).$$

Then, for any $s \in \mathbb{N}$,

$$A = \frac{s}{R^n} \left(\frac{1}{2} - \delta - \nu - 12\delta C_2 \right)$$

is strictly positive, and the sampling estimate

$$(12) \quad A\|f\|_2^2 \leq \sum_{j=1}^s |f(x_j)|^2 \leq sC_1^2\|f\|_2^2, \quad \forall f \in V_{R,\delta}(\Phi)$$

holds with probability at least

$$(13) \quad 1 - R^{n^2/\alpha'} \beta^n \exp\left(-\frac{\nu^2 s}{C_1^2 R^n (1 + \nu/3)}\right) - (R+2)^n \exp\left(-\frac{s}{R^n} (3 \log 3 - 2)\right).$$

Here β is defined by (3).

Proof. Define the random variable N_0 as the covering index of x_1, \dots, x_s . Fix $N = N(R)$, and consider the events

$$V_1 = \left\{ \inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{s} \sum_{j=1}^s (|f(x_j)|^2 - R^{-n} \|Q_R f\|_2^2) \leq -\nu R^{-n} \right\}$$

and

$$V_2 = \{N_0 \geq 3R^{-n}\}.$$

By Lemma 12, we have for all (x_1, \dots, x_s) in the complement of $V_1 \cup V_2$ and all $f \in V_{R,\delta}(\Phi)$,

$$\frac{1}{s} \sum_{j=1}^s |f(x_j)|^2 \geq A\|f\|_2^2,$$

where we used $\gamma = 1/2$ (due to our choice of $N = N(R)$) to simplify A to the constant occurring in (12).

Theorem 10 combined with Lemma 6 yields that V_1^c occurs with probability at most

$$\beta^n R^{n^2/\alpha'} \exp\left(-\frac{\nu^2 s}{C_1^2 R^n (1 + \nu/3)}\right).$$

Furthermore, Lemma 13 yields that V_2^c occurs with probability at most

$$(R+2)^n \exp(-sR^{-n}(3 \log 3 - 2)).$$

Thus the lower estimate in (12) occurs at least with the probability given in (13), whereas the upper estimate follows from the definition of C_1 . \square

Proof of Theorem 2.

The requirements on the various quantities guarantee the applicability of Theorem 14. Furthermore, for $R \geq 1$, we find that by assumptions on R and ν ,

$$\begin{aligned}
 & \frac{(R+2)^n \exp\left(-\frac{s}{R^n}(3\log 3 - 2)\right)}{\beta^n R^{n^2/\alpha'} \exp\left(-\frac{s\nu^2}{R^n C_1^2(1+\nu/3)}\right)} \\
 & \leq \underbrace{\frac{3^n R^{n-n^2/\alpha'}}{\beta^n}}_{\leq 1, \text{ since } \beta > 3} \underbrace{\exp\left(-\frac{s}{R^n}\left(3\log 3 - 2 - \frac{\nu^2}{C_1^2(1+\nu/3)}\right)\right)}_{\leq 1} \\
 (14) \quad & \leq 1.
 \end{aligned}$$

Hence, as soon as

$$s \geq R^n \frac{1 + \nu/3}{\nu^2} \log \frac{2\beta^n R^{n^2/\alpha'}}{\epsilon},$$

the first term subtracted in (13) is $\leq \epsilon/2$, and greater or equal to the second term, by (14). The theorem is proved. \square

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